DEDUCTION OF THE QUANTUM NUMBERS OF LOW-LYING STATES OF 4-NUCLEON SYSTEMS BASED ON SYMMETRY

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Abstract

The inherent nodal structures of the wavefunctions of 4-nucleon systems have been investigated. The existence of two groups of low-lying states with specific quantum numbers dominated by total orbital angular momentum L=0 and L=1, respectively, has been deduced. The understanding of the inherent nodal structure is found to be crucial to a systematic understanding of the spectrum.

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(no figures and four tables)

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In the investigation of microstructures the physicists have paid great attention to the role of symmetry. A number of laws (or constraints) governing the basic physical processes have been obtained [1-4]. However, it is well known that the feature of quantum states depends on the distribution of wavefunctions in the coordinate space, more specially on the nodal structure of the wavefunctions. Will the nodal structure be affected by symmetry? How does symmetry affect the nodal structure if it does? These problems have not yet been studied systematically. Since eigenstates must be orthogonal to each other, nodal surfaces must be introduced in higher states so that they can be orthogonal to lower states. However there exists another kind of nodal surfaces not arising from the requirement of orthogonality but being imposed by the inherent symmetry of the wavefunctions (namely the symmetry with respect to rotation, space inversion, and particle permutation) [5]. This kind of nodal surfaces is called the inherent nodal surface (INS). They are fixed at body-frames and can not be shifted by adjusting dynamical parameters. In this letter, an example of a 4-nucleon system will be used to demonstrate the origin of the INS and the effect of them on low-lying spectra.

A systematic understanding of the energy spectra has in general not yet been obtained for few-body systems, although a number of methods have been developed to solve the Schrödinger equation to obtain precise solutions to explain the observables. For example, it is possible to recover the ordering of levels via a precise theoretical calculation, but it is difficult to explain why a spectrum looks as it is. In order to understand better the physics underlying the spectra, a qualitative study is in general necessary. In particular, an analysis of the inherent nodal structure can uncover why the wavefunction of a specific state is distributed in the coordinate space in a specific way, and why a state with a specific set of quantum numbers is lower or higher(as we shall see).

Let Ψ_{LM} represent an antisymmetrized wavefunction of a 4-nucleon system with total orbital angular momentum L, total spin S, total isospin T, and parity Π. M is the Z-component of L. Ψ_{LM} can be expanded as

$$\Psi_{LM} = \sum_{\lambda} \psi^{\lambda}, \tag{1}$$

where λ denotes the permutation symmetry of the spatial wavefunction (a representation of the S(4) group), ψ^{λ} is called a λ-component of Ψ_{LM}. The λ contained in Ψ_{LM} are determined by S and T [6] as shown in Table 1.

Let i'-j'-k' be a body frame with k' being normal to $\vec{r}_{12}$ and i' parallel to $\vec{r}_{12}$. Then

$$\psi^{\lambda} = \sum_{Q} D^{L}_{QM}(-\gamma,-\beta,-\alpha) \sum_{i} f^{\lambda}_{iQ} \chi^{\tilde{\lambda}}_{i} \tag{2}$$

where D^{L}_{QM} is the Wigner function; α,β and γ are the Euler angles that specify the orientation of the body frame; Q is the component of L along k'; f^{λ}_{iQ} a spatial function of the coordinates relative to the body frame, $\chi^{\tilde{\lambda}}_{i}$ a spin-isospin state, $\tilde{\lambda}$ the conjugate representation of λ . The subscript i labels the basis functions to

span the λ ($\tilde{\lambda}$) representation. It is noted that the f^{λ}_{iQ} span a representation of the rotation group, inversion group, and permutation group. This point is crucial to the following discussion.

When the particles form a shape with a specific geometric symmetry, specific constraints may be imposed on the wavefunction. The greater the geometric symmetry , the stronger the constraints. For example,

(i) If a shape contains a 2-fold axis lying along k' with the particles 1 and 2 (3 and 4) being symmetric to this axis,; i.e., $\vec{r}_{12} \perp k'$ and $\vec{r}_{34} \perp k'$, Then a spatial rotation about k' by 180^0 is equivalent to an interchange of the locations of particles 1 and 2, together with an interchange of 3 and 4. Thus we have

$$(-1)^Q f^{\lambda}_{iQ} = \sum_{i'} g^{\lambda}_{ii'}(p_{12}p_{34}) f^{\lambda}_{i'Q} \tag{3}$$

where the factor $(-1)^Q$ arises from the rotation, and $g^{\lambda}_{ii'}$ is a matrix element of the λ-representation (they are known constants from the theory of the permutation group [7,8]), p_{12} and p_{34} denote the interchange of particles.

(ii) Additionally, if the shape has $\vec{r}_{12} \perp \vec{r}_{34}$ further, a rotation about i' by 180^0 together with a spatial inversion is equivalent to an interchange of 1 and 2. In the rotation f^{λ}_{iQ} is changed to $(-1)^L f^{\lambda}_{i\bar{Q}}$. Thus, in addition to eq.(3), we have

$$\Pi(-1)^L f^{\lambda}_{i\bar{Q}} = \sum_{i'} g^{\lambda}_{ii'}(p_{12}) f^{\lambda}_{i'Q} \ . \tag{4}$$

(iii) Additionally, if the shape has $r_{12} = r_{34}$ further, the shape is now a prolonged (or flattened) regular tetrahedron. Then a rotation about k' by -90^0 together with an inversion is equivalent to the cyclic permutation (1423). Thus, in addition to eq. (3) and (4), we have

$$\Pi i^Q f^{\lambda}_{jQ} = \sum_{j'} g^{\lambda}_{jj'}[(1423)] f^{\lambda}_{j'Q} \ . \tag{5}$$

Eqs.(3) to (5) impose a strong constraint on the wavefunction, the f^{λ}_{jQ} at any prolonged (or flattened) regular tetrahedron must fulfill these equations.

(iv) Additionally, if the height length of the above prolonged tetrahedron is equal to $\frac{r_{12}}{\sqrt{2}}$ (and $\frac{r_{34}}{\sqrt{2}}$), then the shape is an equilateral tetrahedron (ETH). In this case each of the axes along $\vec{r_i}$ (originating from the center of mass) is a 3-fold axis. For example, a spatial rotation about the axis along $\vec{r_1}$ by 120^0 is equivalent to a cyclic permutation of particles 2, 3, and 4. Thus , in addition to eqs. (3) , (4), and (5), we have

$$\sum_{Q'} B_{QQ'} f^{\lambda}_{iQ'} = \sum_{i'} g^{\lambda}_{ii'}[(234)] f^{\lambda}_{i'Q} \tag{6}$$

where

$$B_{QQ'} = \sum_{Q"} D^L_{Q"Q}(0,\Theta,0) e^{-i\frac{2\pi}{3}Q"} D^L_{Q"Q'}(0,\Theta,0) \tag{7}$$

where Θ is the angle between $\vec{r_1}$ and k', cos $\Theta = \sqrt{1/3}$.

The wavefunctions at an ETH must fulfill eqs.(3) to (6).

The above constraints imposed on the shape are sufficient to specify an ETH. For the ETH configuration, the constraints expressed by (3) to (6) are complete, one can prove that other "new" constraints are equivalent constraints. The equations (3) to (6) are homogeneous linear algebra equations depending on L, Π, and λ. It is well known that homoegneous equations do not always have nonzero solutions. Since the search of nonzero solutions of linear equations is trivial, the result is directly given in Table 2. Where, in a few cases (associated with an empty block), there is a set of nonzero solutions f_{iQ}^{λ} satisfying all these equations; it implies that the associated λ-component ψ^{λ} (refer to eq.(2)) is nonzero at the ETH configurations and, we may say, this λ-component is ETH-accessible. In other cases (associated with a block with a $\times$) , there are no nonzero solutions, all the f_{iQ}^{λ} must be zero at ETH configurations irrespective to the size and orientation of the ETH. In such cases, an INS appears and the λ-component is ETH-inaccessible. This example shows the origin of the INS.

The INS existing at the ETH may extend beyond the ETH. For example, when nonzero solutions of (3) to (6) can not be found and nonzero solutions of only (3) to (5) also can not be found, the INS will extend from the ETH to the prolonged (or flattened) regular tetrahedron. Since an ETH has many possibilities to deform (e.g., the height from a vertex to its opposite base becomes longer or shorter), the INS at the ETH has many possibilities to extend to its neighborhood. Thus, there may be a source at the ETH, where the INS emerges and extends to ite neighborhood. In other words, a wavefunction may have an inherent nodal structure in the domain surrounding the ETH. The details of the inherent structure depends on L, Π, and λ of the wavefunction, but does not at all depend on any dynamical parameters. It is emphasized that if a wavefunction can access the ETH, then it can access the neighborhood also. Therefore, the ETH-accessible wavefunction is inherent nodeless in the large domain surrounding the ETH.

Besides the source at the ETH, another source of INS may locate at the squares. When the particles form a square with particles 1 and 2 (3 and 4) at the two ends of a diagonal, we have the following equivalences. (i) a space inversion is equivalent to $p_{12}p_{34}$, (ii) when k' is normal to the plane of the square, a rotation about k' by 180^0 is equivalent to an inversion, (iii) a rotation about i' by 180^0 is equivalent to p_{34}, and (iv) a rotation about k' by -90^0 is equivalent to a cyclic permutation (1324). Similar to the previous discussion, these equivalences impose constraints on the λ-components ψ^{λ} . Thus the accessibility of the square can be thereby identified as given also in Table 2. Similarly, when a wavefunction can access the squares, it is inherent nodeless in the domain surrounding the squares.

There may be other sources of INS (e.g., the one locates at collinear configurations). However, since the total potential energy is much higher in the domains surrounding the other sources, and since we are interested only in low-lying states, we shall neglect the other sources.

Evidently, all the low-lying states tend to contain as least as possible the number

of nodal surfaces , simply because a nodal surface would cause a remarkable increase in energy. Hence, the most important λ-components for the low-lying states should be both ETH-accessible and square-accessible, which are essentially inherent nodeless (observed in the body frame) . They are called inherent-nodeless λ-components. It is noted that, in the higher states, an inherent-nodeless λ-component is allowed to contain additional nodal surfaces due to requirement of orthogonality [5]. Furthermore, it is also allowed to contain nodal surfaces associated with the $\alpha\beta\gamma$ degrees of freedom (i.e., they are permitted to have an excitation of collective rotation as we shall see in the excited states of ^{4}He).

Now it is able to deduce the quantum numbers of low-lying states (resonances) based on an assumption that these states are dominated by inherent-nodeless λ-components. For a 4-nucleon system, since the size is small, the collective energy is large if L is not small. Let us first concentrate on the case of L$\leq$1. From Table 2 it is found that there are three inherent-nodeless λ-components, they have the (L Π λ) equal to (0,+1,$\{4\}$), (1,+1,$\{2,1,1\}$), and (1,-1,$\{3,1\}$) , respectively. Let us see how they compose the low-lying spectrum.

For the states of S=0 and T=0, the $\{4\}$, $\{2,2\}$, and $\{1^4\}$ spatial symmetries are allowed (refer to Table 1). Hence, the (0,+1,$\{4\}$) inherent-nodeless component may be contained. Accordingly, we deduce that there is a $J^{\Pi} = 0^+$ and T = 0 state dominated by this inherent-nodeless component as listed in the second row of Table 3.

For the states of S=1 and T=0, the $\{2,1,1\}$ and $\{3,1\}$ symmetries are allowed. The former may contain the (1,+1,$\{$2,1,1$\}$) inherent-nodeless component, while the latter may contain the (1,-1,$\{$3,1$\}$) inherent-nodeless component. In both cases, the L and S may be coupled to J=0,1, and 2. Thus we deduce that there is an even-parity $J^{\Pi} = 0^+$, 1^+, and 2^+ multiplet, together with an odd-parity $J^{\Pi} = 0^-$, 1^-, and 2^- multiplet. Both multiplets have L=1, as shown in Table 3.

For the states of S=2 and T=0, only the $\{2,2\}$ symmetry is allowed, where no inherent-nodeless components can be contained (if L $\leq$ 1). Thus we deduce that the states of S=2 and T=0 must be higher.

Based on the inherent nodal structure, without solving the Schrödinger equation and without using any dynamical model, we have deduced that there are seven low-lying T=0 states. All the other states of T=0 should be remarkably higher , because either they are dominated by L $\geq$ 2 components, or they do not contain inherent-nodeless λ-components. In Table 3 the J, Π, L, S, and λ of the above predicted states are listed, where the L, S, and λ are only the quantum numbers of the dominant component. Since the 4- nucleon system is small in size, the collective rotation energy with L=1 is large. Thus we predict that there is a large energy gap lying between the state of L=0 (the ground state) and the six higher states of L=1.

All the T=0 levels of ^{4}He below 28.31 MeV (very close to the 2n+2p threshold) from an R-matrix analysis [9] based on experimental data are given in Table 4. As can be seen from this table, there is a big gap lying between the ground state and the six higher states, just as predicted. Moreover, all the values of J^{Π} of these

seven states are exactly the same as predicted. However, it may be necessary to point out that , except the ground state and the first excited 0^+ state, other states as resonances derived from the R-matrix analysis have not yet been commonly recognized. Although our analysis coincides with the analysis of [9], both analyses can not assure the existence of the resonances. The six L=1 states definitely will be split by nuclear force, the details of the split can not be foretold by symmetry. Besides, how the width of a resonance would be affected by symmetry remains open.

Traditionally, the spectrum has been explained based on the shell model [10]. However, the appearance of the 0_2^+ state at 20.21 MeV is not anticipated by the shell model. In this model the 0_2^+ would contain two quanta ($2\hbar\Omega$) of excitation, therefore it would be much higher than the odd parity states containing only $1\hbar\Omega$ of excitation. But in fact the 0_2^+ is lower than all the odd parity states. On the other hand, in our approach the appearance of the 0_2^+ is explained as an excitation of collective rotation (L increases from 0 to 1) together with a change of λ from $\{4\}$ to $\{2,1,1\}$, and a change of S from 0 to 1. It was shown that in the calculation based on shell model multi-$\hbar\Omega$ configurations are absolutely necessary [11]. This fact is explicit because all the excited states are not dominated by the $\{4\}$ component (refer to Table 3), therefore the core-excitation will be very serious and will lead to poor convergency. In fact the weight of the 0-$\hbar\Omega$ component in the T=0 0_2^+ state is only 0.08 (refer to eq.(13) of [11]).

In the same way we can also predict the states of T=1. For odd parity states, the (1,-1,{3.1}) inherent-nodeless λ-component can be contained in [S,T] = [0,1] and [1,1] states (refer to Table 1). In the latter case the S and L will be coupled to J=0,1, and 2. Therefore four odd parity T=1 states are predicted as listed in Table 3. All of them have been found in ^{4}H, ^{4}He (refer to Table 4), and ^{4}Li with exactly the predicted J and Π [9]. Besides, there are also a number of even parity T=1 states containing inherent-nodeless λ-components, but they are higher and do not appear in the low-lying spectrum. This point can not be explained simply from symmetry, but it depends seriously on the feature of nuclear force.

The decisive effect of the inherent nodal structures presents in all kinds of few-body systems. For one more example, if a 4-valence-electron atom (ion) with an inert core is concerned, we have S=0,1,and 2. In accord with S, the λ for the spatial function equal to $\{2,2\}$, $\{2,1,1\}$, and $\{1^4\}$, respectively. Thus, among the three inherent-nodeless λ-components mentioned above, only the (1,+1,$\{2,1,1\}$) is available (if L $\leq$ 1). Thus the ground state must not be a state of L=0, but a $^3P^e$ state. This fact is confirmed by all the experimental data of the 4-valence-electron atoms (ions) including the C, N^+ , O^{++} , F^{+++} , Si, P^+ , S^{++} , Ge, etc.

When a 4-boson system is concerned, only the $\{4\}$ symmetry is involved. Let us consider the ^{16}O nucleus as a system of four α-particles. Since this system has a larger size and a heavier mass (as compared to the ^{4}He), the states with a larger L may also appear in the low-lying spectrum. When all the L $\leq$ 4 states are concerned, three and only three of them, namely the 0^+ , 3^- , and 4^+, can access the ETH. Let us define the internal energy of a state as E_I = E - E_{rot} , where E_{rot}

is the rotation energy. When E_{rot} is roughly estimated via an evaluation of the moment of inertia, we find that the E_I of the 0^+ , 3^- , and 4^+ states are very close to each other, and they are remarkably lower than the E_I of the other states. [12]

The ideas and theoretical procedures proposed in this letter can be directly generalized to study other few-body systems. In any case, the inherent nodal structure in the domains surrounding regular shapes (e.g., a regular octahedron in the case of a 6-body system) should be investigated [13]. The existence of the inherent nodal structures in microscopic few-body systems is a great marvel of quantum mechanics. Since the INS are decisive and they are common to different systems, similarity exists surely among these systems. Through an investigation of the INS, different systems can be recognized via an unified point of view.

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S	T	λ
0	0	{4},{2,2},{1^4}
1	0	{3,1},{2,1,1}
2	0	{2,2}
0	1	{3,1},{2,1,1}
1	1	{3,1},{2,2},{2,1,1},{1^4}
2	1	{2,1,1}

Table 1

		{4}	{3,1}	{2,2}	{2,1,1}	{1^4}
0^+	ETH		$\times$	$\times$	$\times$	$\times$
0^+	square		$\times$		$\times$	$\times$
0^-	ETH	$\times$	$\times$	$\times$	$\times$	
0^-	square	$\times$	$\times$	$\times$	$\times$	$\times$
1^+	ETH	$\times$	$\times$	$\times$		$\times$
1^+	square	$\times$	$\times$	$\times$		$\times$
1^-	ETH	$\times$		$\times$	$\times$	$\times$
1^-	square	$\times$		$\times$		$\times$

Table 2

T	J^Π	L	S	λ
0	0^+	0	0	{4}
0	0^-	1	1	{3,1}
0	1^-	1	1	{3,1}
0	2^-	1	1	{3,1}
0	0^+	1	1	{2,1,1}
0	1^+	1	1	{2,1,1}
0	2^+	1	1	{2,1,1}
1	0^-	1	1	{3,1}
1	1^-	1	1	{3,1}
1	2^-	1	1	{3,1}
1	1^-	1	0	{3,1}

Table 3

T	J^{Π}	MeV	T	J^{Π}	MeV
0	0^+	0	1	2^-	23.3
0	0^+	20.2	1	1^-	23.6
0	0^-	21.0	1	0^-	25.3
0	2^-	21.8	1	1^-	26.0
0	1^-	24.3			
0	2^+	27.4			
0	1^+	28.3			

Table 4

Table Captions

Table 1 The allowed representation λ of the spatial wavefunctions. $\{4\}$ denotes the totally-symmetric symmetry,etc.

Table 2 The accessibility of the ETH (equilateral tetrahedron) and the square configurations to the $(L\Pi\lambda)$ wavefunctions. The first column is L^{Π}, the first row is λ. An empty block implies being accessible, a block with a $\times$ implies being inaccessible.

Table 3 The predicted quantum numbers of low-lying states ($L \leq 1$). Each of them contains an inherent-nodeless λ-components. λ is the permutation symmetry of the spatial wavefunctions.

Table 4, Low-lying spectrum of ^4He nucleus from an R-matrix analysis of experimental data [9].